

ON THE MOD-2 COHOMOLOGY OF $SL_3(\mathbb{Z}[\frac{1}{2}, i])$

HANS-WERNER HENN

ABSTRACT. Let $\Gamma = SL_3(\mathbb{Z}[\frac{1}{2}, i])$, let X be any mod-2 acyclic Γ -CW complex on which Γ acts with finite stabilizers and let X_s be the 2-singular locus of X . We calculate the mod-2 cohomology of the Borel construction of X_s with respect to the action of Γ . This cohomology coincides with the mod-2 cohomology of Γ in cohomological degrees bigger than 8 and the result is compatible with a conjecture of Quillen which predicts the structure of the cohomology ring $H^*(\Gamma; \mathbb{Z}/2)$.

1. INTRODUCTION

The mod-2 cohomology of the group $SL_3(\mathbb{Z}[\frac{1}{2}, i])$ can be approached in the same way as that of $SL_3(\mathbb{Z}[\frac{1}{2}])$. In a first step one uses a centralizer spectral sequence introduced in [H1] in order to calculate the mod-2 cohomology of the Borel cohomology $H_G^*(X_s; \mathbb{F}_2)$ where X is any mod-2 acyclic G -CW complex on which a given discrete group G acts with finite stabilizers and X_s is the 2-singular locus of X , i.e. the subcomplex consisting of all points for which the isotropy group of the action of G is of even order. For $G = SL_3(\mathbb{Z}[\frac{1}{2}])$ this step was carried out in [H1] and for $G = SL_3(\mathbb{Z}[\frac{1}{2}, i])$ it is carried out in this paper. The precise form of X does not really matter in this step.

The second step involves a very laborious analysis of the relative Borel cohomology $H_G^*(X, X_s; \mathbb{F}_2)$ and of the connecting homomorphism for the Borel cohomology of the pair (X, X_s) . In the case of $G = SL_3(\mathbb{Z}[\frac{1}{2}])$ this was carried out by hand in [H2]. A by hand calculation looks forbidding in the case of $G = SL_3(\mathbb{Z}[\frac{1}{2}, i])$ and this paper makes no attempt on such a calculation. However, we do make some comments on what is likely to be involved in such an attempt.

Here are the main results of this paper. In these results the elements v_2 respectively v_3 are of degree 4 resp. 6. They are related to the Chern classes of the tautological 3 dimensional complex representation of $SL_3(\mathbb{Z}[\frac{1}{2}, i])$. The indices of the other elements give their cohomological degrees. These elements come from Quillen's exterior cohomology classes in the cohomology of $GL_3(\mathbb{F}_p)$ for suitable primes p , for example for $p = 5$ (cf. section 5 for more details). Furthermore Σ^n denotes n -fold suspension so that $\Sigma^4 \mathbb{F}_2$ is a one dimensional \mathbb{F}_2 -vector space concentrated in degree 4.

Theorem 1.1. *Let $\Gamma = SL_3(\mathbb{Z}[\frac{1}{2}, i])$ and let X be any mod 2-acyclic Γ -CW complex such that the isotropy group of each cell is finite. Then the centralizer spectral sequence of [H1]*

$$\lim_{\mathcal{A}_*(\Gamma)}^s H^t C_\Gamma(E; \mathbb{F}_2) \implies H_\Gamma^{s+t}(X_s; \mathbb{F}_2)$$

collapses at E_2 and gives a short exact sequence

$$0 \rightarrow \Sigma^4 \mathbb{F}_2 \oplus \Sigma^4 \mathbb{F}_2 \oplus \Sigma^7 \mathbb{F}_2 \rightarrow H_\Gamma^*(X_s; \mathbb{F}_2) \rightarrow \mathbb{F}_2[v_2, v_3] \otimes \Lambda(d_3, d'_3, d_5, d'_5) \rightarrow 0$$

in which the second map is a map of graded algebras.

Theorem 1.2. *Let $\Gamma = SL_3(\mathbb{Z}[\frac{1}{2}], i)$ and X be as in the previous theorem.*

a) If $SD_3(\mathbb{Z}[\frac{1}{2}], i)$ denotes the subgroup of diagonal matrices of Γ then the restriction homomorphism $H^(B\Gamma; \mathbb{Z}/2) \rightarrow H^*(SD_3(\mathbb{Z}[\frac{1}{2}], i); \mathbb{F}_2)$ coincides with the composition*

$$\psi : H^*(\Gamma; \mathbb{F}_2) = H_\Gamma^* X \rightarrow H_\Gamma^*(X_s; \mathbb{F}_2) \rightarrow \mathbb{F}_2[v_2, v_3] \otimes \Lambda(d_3, d'_3, d_5, d'_5)$$

of the map induced by the inclusion $X_s \subset X$ and the epimorphism of Theorem 1.1.

b) There exists a map of graded \mathbb{F}_2 -algebras

$$\varphi : \mathbb{F}_2[c_2, c_3] \otimes \Lambda(e_3, e'_3, e_5, e'_5) \rightarrow H^*(\Gamma; \mathbb{F}_2)$$

such that its composition with

$$\psi : H^*(\Gamma; \mathbb{F}_2) = H_\Gamma^* X \rightarrow H_\Gamma^*(X_s; \mathbb{F}_2) \rightarrow \mathbb{F}_2[v_2, v_3] \otimes \Lambda(d_3, d'_3, d_5, d'_5)$$

sends c_i to v_i , $i = 2, 3$, e_i to d_i and e'_i to d'_i , $i = 3, 5$.

c) The homomorphism ψ is surjective in all degrees, an isomorphism in degrees $ > 8$ and the kernel is finite dimensional in degrees $* \leq 8$.*

Remark 1.3. In section 5 we will discuss the relation of Theorem 1.2 with a conjecture of Quillen on the structure of the cohomology of $H^*(GL_n(\Gamma); \mathbb{F}_2)$ (cf. 14.7 of [Q1]) which would hold in case $n = 3$ if the composition in part (b) of Theorem 1.2 turned out to be an isomorphism.

The following result is an immediate consequence of Theorem 1.2.

Corollary 1.4. *Let $\Gamma = SL_3(\mathbb{Z}[\frac{1}{2}], i)$ and X be as in Theorem 1.1. Then the following conditions are equivalent.*

a) The restriction homomorphism $H^(B\Gamma; \mathbb{Z}/2) \rightarrow H^*(SD_3(\mathbb{Z}[\frac{1}{2}], i); \mathbb{F}_2)$ is an isomorphism.*

b) There is an isomorphism

$$H_\Gamma^*(X, X_s; \mathbb{F}_2) \cong \Sigma^5 \mathbb{F}_2 \oplus \Sigma^5 \mathbb{F}_2 \oplus \Sigma^8 \mathbb{F}_2$$

and the connecting homomorphism $H_\Gamma^(X_s) \rightarrow H_\Gamma^{*+1}(X, X_s)$ is surjective.* \square

The paper is organized as follows. In section 2 we recall the centralizer spectral sequence and in section 3 we prove Theorem 1.1. and Theorem 1.2. In Section 4 we make some comments on step 2 of the program of a complete calculation of $H^*(\Gamma; \mathbb{F}_2)$. Finally in section 5 we discuss the relation with Quillen's conjecture.

2. THE CENTRALIZER SPECTRAL SEQUENCE

For the convenience of the reader we recall the centralizer spectral sequence introduced in [H1].

Let G be a discrete group and let p be a fixed prime. Let $\mathcal{A}(G)$ be the category whose objects are the elementary abelian p -subgroups E of G , i.e. subgroups which are isomorphic to $(\mathbb{Z}/p)^k$ for some integer k ; if E_1 and E_2 are elementary abelian p -subgroups of G , then the set of morphisms from E_1 to E_2 in $\mathcal{A}(G)$ consists precisely of those group homomorphisms $\alpha : E_1 \rightarrow E_2$ for which there exists an element $g \in G$ with $\alpha(e) = geg^{-1}$ for all $e \in E_1$. Let $\mathcal{A}_*(G)$ be the full subcategory of $\mathcal{A}(G)$ whose objects are the nontrivial elementary abelian p -subgroups.

For an elementary abelian p -subgroup we denote its centralizer in G by $C_G(E)$. Then the assignment $E \mapsto H^*(C_G(E); \mathbb{Z}/p)$ determines a functor from $\mathcal{A}_*(G)$ to the category \mathcal{E} of graded \mathbb{F}_p vector spaces. The inverse limit functor is a left exact functor from the functor category $\mathcal{E}^{\mathcal{A}_*(G)}$ to \mathcal{E} . Its right derived functors are denoted by \lim^s . The p -rank $r_p(G)$ of a group G is defined as the supremum of all k such that G contains a subgroup isomorphic to $(\mathbb{Z}/p)^k$.

For a G -space X and a fixed prime p we denote by X_s the p -singular locus, i.e. the subspace of X consisting of points whose isotropy group contains an element of order p . Let EG be the total space of the universal principal G -bundle. The mod- p cohomology of the Borel construction $EG \times_G X$ of a G space X will be denoted $H_G^*(X)$. The following result is a special case of part (a) of Corollary 0.4 of [H1].

Theorem 2.1. *Let G be a discrete group and assume there exists a finite dimensional mod- p acyclic G -CW complex X such that the isotropy group of each cell is finite. Then there is a cohomological spectral sequence*

$$E_2^{s,t} = \lim_{\mathcal{A}_*(G)}^s H^t(C_G(E); \mathbb{F}_p) \implies H_G^{s+t}(X_s; \mathbb{F}_p)$$

with $E_2^{s,t} = 0$ if $s \geq r_p(G)$ and $t \geq 0$.

In [H1] we have used this spectral sequence in the case $p = 2$ and $G = SL_3(\mathbb{Z})$. Here we will use it in the case $p = 2$ and $G = SL(3, \mathbb{Z}[\frac{1}{2}, i])$. In both cases we have $r_2(G) = 2$ and hence the spectral sequence collapses at E_2 and degenerates into a short exact sequence

$$(2.1) \quad 0 \rightarrow \lim_{\mathcal{A}_*(G)}^1 H^t(C_G(E); \mathbb{F}_p) \rightarrow H_G^{t+1}(X_s; \mathbb{F}_p) \rightarrow \lim_{\mathcal{A}_*(G)} H^{t+1}(C_G(E); \mathbb{F}_p) .$$

3. THE CENTRALIZER SPECTRAL SEQUENCE FOR $SL_3(\mathbb{Z}[\frac{1}{2}, i])$

We start by describing the Quillen category of $\Gamma := SL_3(\mathbb{Z}[\frac{1}{2}, i])$ for $p = 2$ and the functor which sends E to $H^*(C_\Gamma(E); \mathbb{F}_2)$.

3.1. The Quillen category. Let K be any number field, let \mathcal{O}_K be its ring of integers and consider the ring of S -integers $\mathcal{O}_K[\frac{1}{2}]$. Then, up to equivalence, the Quillen category of $G := SL_3(\mathcal{O}_K[\frac{1}{2}])$ for the prime 2 is independant of K . In fact, because 2 is invertible every elementary abelian 2- subgroup is conjugate to a diagonal subgroup, and hence $\mathcal{A}_*(G)$ has a skeleton, say \mathcal{A} , with exactly two objects, say E_1 and E_2 of rank 1 and 2, respectively. We take E_1 to be the subgroup generated by the diagonal matrix whose first two diagonal entries are -1 and whose third diagonal entry is 1, and E_2 to be the subgroup of all diagonal matrices with diagonal entries 1 or -1 and determinant 1.

The automorphism group of E_1 is trivial, of course, while $\text{Aut}_{\mathcal{A}}(E_2)$ is isomorphic to the group of all abstract automorphisms of E_2 which we can identify with \mathfrak{S}_3 , the symmetric group on three elements. There are three morphisms from E_1 to E_2 and $\text{Aut}_{\mathcal{A}}(E_2)$ acts transitively on them.

3.2. The centralizers and their cohomology. For the centralizers in $H := GL_3(\mathcal{O}_K[\frac{1}{2}])$ we find $C_H(E_1) \cong GL_2(\mathcal{O}_K[\frac{1}{2}]) \times GL_1(\mathcal{O}_K[\frac{1}{2}])$ resp. $C_H(E_2) \cong D_3(\mathcal{O}_K[\frac{1}{2}])$ the subgroup of diagonal matrices which is isomorphic to $\prod_{i=1}^3 GL_1(\mathcal{O}_K[\frac{1}{2}])$. This implies that $C_G(E_1) \cong GL_2(\mathcal{O}_K[\frac{1}{2}])$ and $C_G(E_2) \cong GL_1(\mathcal{O}_K[\frac{1}{2}]) \times GL_1(\mathcal{O}_K[\frac{1}{2}])$.

From now on we specialize to the case $K = \mathbb{Q}_2[i]$ where we have $\mathcal{O}_K[\frac{1}{2}] = \mathbb{Z}[\frac{1}{2}, i]$. In this case the cohomology of the centralizers is explicetely known. In fact, there is an isomorphism

$$\mathbb{Z}/4 \times \mathbb{Z} \cong GL_1(\mathbb{Z}[\frac{1}{2}, i]), \quad (n, m) \mapsto i^n(1+i)^m$$

and therefore we get an isomorphism

$$(3.1) \quad H^*(C_\Gamma(E_2); \mathbb{F}_2) \cong H^*(GL_1(\mathbb{Z}[\frac{1}{2}, i]) \times GL_1(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2) \cong \mathbb{F}_2[y_1, y_2] \otimes \Lambda(x_1, x'_1, x_2, x'_2)$$

with y_1 and y_2 in degree 2 and the other generators in degree 1. We agree to choose the generators so that y_1, x_1 and x'_1 come from the first factor with x_1 and x'_1 being the dual basis to the basis of

$$H_1(GL_1(\mathbb{Z}[\frac{1}{2}, i]; \mathbb{F}_2) \cong \mathbb{Z}[\frac{1}{2}, i]^\times / (\mathbb{Z}[\frac{1}{2}, i]^\times)^2$$

given by the image of i and $(1+i)$ in the mod-2 reduction of the abelian group $GL_1(\mathbb{Z}[\frac{1}{2}, i])$; likewise with y_2, x_2 and x'_2 coming from the second factor.

Furthermore, from Theorem 1 of [W] we know

$$(3.2) \quad H^*(C_\Gamma(E_1); \mathbb{F}_2) \cong H^*(GL_2(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2) \cong \mathbb{F}_2[c_1, c_2] \otimes \Lambda(e_1, e'_1, e_3, e'_3).$$

In the sequel we give a short summary of his main result. The classes e_1, e'_1, e_3 and e'_3 are pulled back from Quillen's exterior classes q_1 and q_3 [Q2] in

$$H^*(GL_2(\mathbb{F}_5); \mathbb{F}_2) \cong \mathbb{F}_2[c_1, c_2] \otimes \Lambda(q_1, q_3)$$

via the two ring homomorphisms

$$(3.3) \quad \pi_k : \mathbb{Z}[\frac{1}{2}, i] \rightarrow \mathbb{F}_5,$$

$k = 1, 2$, where π_1 sends i to 3 and π_2 sends i to 2. The classes c_1 and c_2 can be also obtained by pulling back, and pulling back via π_1^* or via π_2^* gives the same classes because both restrict to the same elements in $H^*(GL_1(\mathbb{Z}[\frac{1}{2}, i]) \times GL_1(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2)$, namely $y_1 + y_2$ respectively $y_1 y_2$ (see (3.6) below) and Theorem 1 of [W] implies that this restriction is injective.

In fact, in the cohomology of $GL_2(\mathbb{F}_5)$ the classes c_1, c_2, q_1 and q_3 are detected by restriction to the cohomology of diagonal matrices

$$H^*(GL_1(\mathbb{F}_5) \times GL_1(\mathbb{F}_5); \mathbb{F}_2) \cong \mathbb{F}_2[y_1, y_2] \otimes \Lambda(x_1, x_2)$$

$$(3.4) \quad c_1 \mapsto y_1 + y_2, \quad c_2 \mapsto y_1 y_2, \quad q_1 \mapsto x_1 + x_2 \quad q_3 \mapsto y_1 x_2 + y_2 x_1.$$

In particular, if (by some abuse of notation)

$$(3.5) \quad c_1 = \pi_1^*(c_1), \quad c_2 = \pi_1^*(c_2), \quad e_1 = \pi_1^*(q_1), \quad e_3 = \pi_1^*(q_3), \quad e'_1 = \pi_2^*(q_1), \quad e'_3 = \pi_2^*(q_3)$$

then the restriction homomorphism from $H^*(GL_2(\mathbb{Z}[\frac{1}{2}]); \mathbb{F}_2)$ to the cohomology of the subgroup of diagonal matrices is given by

$$(3.6) \quad \begin{array}{ll} c_1 \mapsto y_1 + y_2 & c_2 \mapsto y_1 y_2 \\ e_1 \mapsto x_1 + x_2 & e_3 \mapsto y_1 x_2 + y_2 x_1 \\ e'_1 \mapsto x_1 + x'_1 + x_2 + x'_2 & e'_3 \mapsto y_1(x_2 + x'_2) + y_2(x_1 + x'_1) \end{array}$$

and the restriction of $\pi_1^*(c_1)$ and $\pi_2^*(c_1)$ respectively $\pi_1^*(c_2)$ and $\pi_2^*(c_2)$ agree. We also see from (3.6) that the restriction homomorphism for $H^*(GL_2(\mathbb{Z}[\frac{1}{2}]); \mathbb{F}_2)$ is injective and therefore $\pi_1^*(c_i) = \pi_2^*(c_i)$ for $i = 1, 2$.

Furthermore we note that together with the isomorphisms (3.1) and (3.2) this restriction also describes the map

$$\alpha_* : H^*(C_\Gamma(E_1); \mathbb{F}_2) \rightarrow H^*(C_\Gamma(E_2); \mathbb{F}_2)$$

induced from the standard inclusion of E_1 into E_2 .

To finish the description of $H^*C_\Gamma(-; \mathbb{F}_2)$ as a functor on \mathcal{A} it remains to describe the action of $\text{Aut}_{\mathcal{A}}(E_2) \cong \mathfrak{S}_3$ on $H^*(C_\Gamma(E_2); \mathbb{F}_2) \cong \mathbb{F}_2[y_1, y_2] \otimes \Lambda(x_1, x'_1, x_2, x'_2)$ and because of the multiplicative structure we need it only on the generators.

If $\tau \in \text{Aut}_{\mathcal{A}}(E_2)$ corresponds to permuting the factors in $C_\Gamma(E_2) \cong GL_1(\mathbb{Z}[\frac{1}{2}, i]) \times GL_1(\mathbb{Z}[\frac{1}{2}, i])$ then

$$(3.7) \quad \begin{aligned} \tau_*(y_1) &= y_2 & \tau_*(x_1) &= x_2 & \tau_*(x'_1) &= x'_2 \\ \tau_*(y_2) &= y_1 & \tau_*(x_2) &= x_1 & \tau_*(x'_2) &= x'_1 \end{aligned}$$

and if $\sigma \in \text{Aut}_{\mathcal{A}}(E_2)$ corresponds to the cyclic permutation of the diagonal entries (in the appropriate order) then

$$(3.8) \quad \begin{aligned} \sigma_*(y_1) &= y_2 & \sigma_*(x_1) &= x_2 & \sigma_*(x'_1) &= x'_2 \\ \sigma_*(y_2) &= y_1 + y_2 & \sigma_*(x_2) &= x_1 + x_2 & \sigma_*(x'_2) &= x'_1 + x'_2 . \end{aligned}$$

3.3. Calculating the limit and its derived functors. In Proposition 4.3 of [H1] we showed that for any functor F from \mathcal{A} to $\mathbb{Z}_{(2)}$ -modules there is an exact sequence

$$(3.9) \quad 0 \rightarrow \lim_{\mathcal{A}} F \rightarrow F(E_1) \xrightarrow{\varphi} \text{Hom}_{\mathbb{Z}[\mathfrak{S}_3]}(St_{\mathbb{Z}}, F(E_2)) \rightarrow \lim_{\mathcal{A}}^1 F \rightarrow 0$$

where $St_{\mathbb{Z}}$ is the $\mathbb{Z}[\mathfrak{S}_3]$ module given by the kernel of the augmentation $\mathbb{Z}[\mathfrak{S}_3/\mathfrak{S}_2] \rightarrow \mathbb{Z}$, and if a and b are chosen to give an integral basis of $St_{\mathbb{Z}}$ on which τ and σ act via

$$(3.10) \quad \begin{aligned} \tau_*(a) &= b & \tau_*(b) &= a \\ \sigma_*(a) &= -b & \sigma_*(b) &= a - b \end{aligned}$$

then φ is given via $\varphi(x)(a) = \alpha_*(x) - (\sigma_*)^2 \alpha_*(x)$ and $\varphi(x)(b) = \alpha_*(x) - \sigma_* \alpha_*(x)$.

Because in our case the functor takes values in \mathbb{F}_2 -vector spaces we can replace $\text{Hom}_{\mathbb{Z}[\mathfrak{S}_3]}$ by $\text{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}$ and $St_{\mathbb{Z}}$ by its mod-2 reduction. The following elementary lemma is needed in the analysis of the third term in the exact sequence (3.9).

Lemma 3.1.

a) Let St be the $\mathbb{F}_2[\mathfrak{S}_3]$ -module given as the kernel of the augmentation $\mathbb{F}_2[\mathfrak{S}_3/\mathfrak{S}_2] \rightarrow \mathbb{F}_2$. The tensor product $St \otimes St$ decomposes as $\mathbb{F}_2[\mathfrak{S}_3]$ -module canonically as

$$St \otimes St \cong \mathbb{F}_2[\mathfrak{S}_3/A_3] \oplus St$$

where A_3 denotes the alternating group on three letters. In fact, the decomposition is given by

$$St \otimes St \cong \text{Im}(id + \sigma_* + \sigma_*^2) \oplus \text{Ker}(id + \sigma_* + \sigma_*^2)$$

and the first summand is isomorphic to $\mathbb{F}_2[\mathfrak{S}_3/A_3]$ while the second factor is isomorphic to St .

b) The tensor product $\mathbb{F}_2[\mathfrak{S}_3/A_3] \otimes St$ is isomorphic to $St \oplus St$.

Proof. a) It is well known that St is a projective $\mathbb{F}_2[\mathfrak{S}_3]$ -module, hence $St \otimes St$ is also projective. It is also well known that every projective indecomposable $\mathbb{F}_2[\mathfrak{S}_3]$ -module is isomorphic to either St or $\mathbb{F}_2[\mathfrak{S}_3/A_3]$. Both modules can be distinguished by the fact that $e := id + \sigma_* + \sigma_*^2$ acts trivially on St and as the identity on $\mathbb{F}_2[\mathfrak{S}_3/A_3]$.

Furthermore e is a central idempotent in $\mathbb{F}_2[\mathfrak{S}_3]$ and hence each $\mathbb{F}_2[\mathfrak{S}_3]$ -module M decomposes as direct sum of $\mathbb{F}_2[\mathfrak{S}_3]$ -modules

$$M \cong \text{Im}(e : M \rightarrow M) \oplus \text{Ker}(e : M \rightarrow M) .$$

An easy calculation shows that in the case of $St \otimes St$ both submodules are non-trivial and this together with the fact these submodules must be projective proves the claim.

b) Again each of the factors in the tensor product is a projective $\mathbb{F}_2[\mathfrak{S}_3]$ -module, hence the tensor product is a projective $\mathbb{F}_2[\mathfrak{S}_3]$ -module. Because σ acts trivially on $\mathbb{F}_2[\mathfrak{S}_3/A_3]$ we see that the idempotent e acts trivially on the tensor product and this forces the tensor product to be isomorphic to $St \oplus St$. \square

Lemma 3.2. *The Poincaré series χ_2 of $\text{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}(St, \mathbb{F}_2[y_1, y_2] \otimes \Lambda(x_1, x'_1, x_2, x'_2))$ is given by*

$$\chi_2 = \frac{2t^2(1 + 3t^2 + 3t^4 + t^6) + 2t(1 + 2t^2 + 2t^4 + 2t^6 + t^8)}{(1 - t^4)(1 - t^6)}.$$

Proof. The isomorphism of (3.1) is an isomorphism of $\mathbb{F}_2[\mathfrak{S}_3]$ -modules where the action of \mathfrak{S}_3 is given (3.7) and (3.8). In particular we see that $H^*(GL_1(\mathbb{Z}[\frac{1}{2}, i]) \times GL_1(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2)$ is isomorphic to $St \oplus St$ and the exterior powers of H^1 are given as

$$\Lambda^k(x_1, x_2, x'_1, x'_2) \cong \Lambda^k(St \oplus St) \cong \bigoplus_{j=0}^k \Lambda^j St \otimes \Lambda^{k-j} St$$

and hence

$$\Lambda^k(x_1, x_2, x'_1, x'_2) \cong \begin{cases} \Sigma^k \mathbb{F}_2 & k = 0, 4 \\ \Sigma^k(St \oplus St) & k = 1, 3 \\ \Sigma^2 \mathbb{F}_2 \oplus \Sigma^2(St \otimes St) \oplus \Sigma^2 \mathbb{F}_2 & k = 2 \\ 0 & k \neq 0, 1, 2, 3, 4 \end{cases}$$

where \mathbb{F}_2 denotes the (necessarily) trivial $\mathbb{F}_2[\mathfrak{S}_3]$ -module whose additive structure is that of \mathbb{F}_2 . Therefore the Poincaré series χ_2 of $\text{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}(St, H^*(C_G(E_2); \mathbb{F}_2))$ decomposes according to the decomposition of $\Lambda(x_1, x'_1, x_2, x'_2)$ as sum

$$(3.11) \quad \chi_2 := (1 + 2t^2 + t^4)\chi_{2,0} + t^2\chi_{2,1} + 2(t + t^3)\chi_{2,2}$$

where $\chi_{2,0}$ is the Poincaré series of $\text{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}(St, \mathbb{F}_2[y_1, y_2])$, $\chi_{2,1}$ is the Poincaré series of $\text{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}(St, St \otimes St \otimes \mathbb{F}_2[y_1, y_2])$ and $\chi_{2,2}$ is that of $\text{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}(St, St \otimes \mathbb{F}_2[y_1, y_2])$. It is well known (and elementary to verify) that there is an isomorphism of $\mathbb{F}_2[\mathfrak{S}_3]$ modules $St \oplus St \oplus \mathbb{F}_2[\mathfrak{S}_3/A_3] \cong \mathbb{F}_2[\mathfrak{S}_3]$ and therefore an isomorphism

$$\begin{aligned} \mathbb{F}_2[y_1, y_2] &\cong \text{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}(St \oplus St \oplus \mathbb{F}_2[\mathfrak{S}_3/A_3], \mathbb{F}_2[y_1, y_2]) \\ &\cong \text{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}(St, \mathbb{F}_2[y_1, y_2])^{\oplus 2} \oplus \mathbb{F}_2[y_1, y_2]^{A_3}. \end{aligned}$$

Together with the fact that the A_3 -invariants $\mathbb{F}_2[y_1, y_2]^{A_3}$ form a free module over $\mathbb{F}_2[v_2, v_3]$ on two generators of degree 0 resp. 6 this implies

$$2\chi_{2,0} + \frac{1 + t^6}{(1 - t^4)(1 - t^6)} = \frac{1}{(1 - t^2)^2}$$

and hence

$$(3.12) \quad \chi_{2,0} = \frac{t^2}{(1 - t^2)(1 - t^6)}.$$

Furthermore St and $\mathbb{F}_2[\mathfrak{S}_3/A_3]$ are both self-dual $\mathbb{F}_2[\mathfrak{S}_3]$ -modules and hence Lemma 3.1 gives

$$St \otimes St^* \cong St \oplus \mathbb{F}_2[\mathfrak{S}_3/A_3]$$

and

$$\begin{aligned} St \otimes St^* \otimes St^* &\cong (\mathbb{F}_2[\mathfrak{S}_3/A_3] \oplus St) \otimes St^* \\ &\cong (\mathbb{F}_2[\mathfrak{S}_3/A_3] \otimes St) \oplus (St \otimes St) \\ &\cong St \oplus St \oplus St \oplus \mathbb{F}_2[\mathfrak{S}_3/A_3] . \end{aligned}$$

Therefore, if $\chi_{\mathbb{F}_2[y_1, y_2]^{A_3}}$ denotes the Poincaré series of the A_3 -invariants then

$$(3.13) \quad \chi_{2,1} = 3\chi_{2,0} + \chi_{\mathbb{F}_2[y_1, y_2]^{A_3}} = \frac{3t^2}{(1-t^2)(1-t^6)} + \frac{1+t^6}{(1-t^4)(1-t^6)} = \frac{1+3t^2+3t^4+t^6}{(1-t^4)(1-t^6)}$$

$$(3.14) \quad \chi_{2,2} = \chi_{2,0} + \chi_{\mathbb{F}_2[y_1, y_2]^{A_3}} = \frac{t^2}{(1-t^2)(1-t^6)} + \frac{1+t^6}{(1-t^4)(1-t^6)} = \frac{1+t^2+t^4+t^6}{(1-t^4)(1-t^6)} .$$

Finally (3.11), (3.12), (3.13) and (3.14) give

$$\begin{aligned} \chi_2 &= \frac{(1+2t^2+t^4)t^2(1+t^2) + t^2(1+3t^2+3t^4+t^6) + 2(t+t^3)(1+t^2+t^4+t^6)}{(1-t^4)(1-t^6)} \\ &= \frac{2t^2(1+3t^2+3t^4+t^6) + 2t(1+2t^2+2t^4+2t^6+t^8)}{(1-t^4)(1-t^6)} , \end{aligned}$$

and this finishes the proof. \square

Theorem 1.1 is an immediate consequence of Theorem 2.1 and the following result.

Proposition 3.3. *Let $p = 2$ and $\Gamma = SL_3(\mathbb{Z}[\frac{1}{2}, i])$.*

a) *There is an isomorphism of graded \mathbb{F}_2 -algebras*

$$\lim_{\mathcal{A}_*(\Gamma)} H^*(C_\Gamma(E); \mathbb{F}_2) \cong \mathbb{F}_2[v_2, v_3] \otimes \Lambda(d_3, d'_3, d_5, d'_5) .$$

Furthermore, if we identify this limit with a subalgebra of $H^*(C_\Gamma(E_1); \mathbb{F}_2) \cong \mathbb{F}_2[c_1, c_2] \otimes \Lambda(e_1, e'_1, e_3, e'_3)$ then

$$\begin{aligned} v_2 &= c_1^2 + c_2, & v_3 &= c_1 c_2 \\ d_3 &= e_3, & d_5 &= c_1 e_3 + c_2 e_1 \\ d'_3 &= e'_3, & d'_5 &= c_1 e'_3 + c_2 e'_1 . \end{aligned}$$

b) *There is an isomorphism of graded \mathbb{F}_2 -vector spaces*

$$\lim_{\mathcal{A}_*(\Gamma)}^1 H^*(C_\Gamma(E); \mathbb{F}_2) \cong \Sigma^3 \mathbb{F}_2 \oplus \Sigma^3 \mathbb{F}_2 \oplus \Sigma^6 \mathbb{F}_2 .$$

c) *For any $s > 1$*

$$\lim_{\mathcal{A}_*(\Gamma)}^s H^*(C_\Gamma(E); \mathbb{F}_2) = 0 .$$

Proof. a) Via the exact sequence (3.9) and (3.8) it is straightforward to check that the elements v_2, v_3 and d_3, d'_3, d_5, d'_5 are in the inverse limit. Furthermore the subalgebra generated by them is isomorphic to the tensor product of the polynomial algebra $\mathbb{F}_2[v_2, v_3]$ with the exterior algebra $\Lambda(d_3, d'_3, d_5, d'_5)$. In fact, it is clear that v_2 and v_3 are algebraically independant and the elements d_3, d'_3, d_5, d'_5 are exterior classes; their product is given as $c_2^2 e_3 e'_3 e_1 e'_1 \neq 0$, and this implies easily that the exterior monomials in the d 's are linearly independant over $\mathbb{F}_2[v_2, v_3]$.

Now consider the following Poincaré series

$$\begin{aligned}\chi_0 &:= \sum_{n \geq 0} \dim_{\mathbb{F}_2} (\mathbb{F}_2[v_2, v_3] \otimes \Lambda(e_3, e'_3, e_5, e'_5)^n) t^n = \frac{(1+t^3)^2(1+t^5)^2}{(1-t^4)(1-t^6)} \\ \chi_1 &:= \sum_{n \geq 0} \dim_{\mathbb{F}_2} H^n C_G(E_1); \mathbb{F}_2) t^n = \frac{(1+t)^2(1+t^3)^2}{(1-t^2)(1-t^4)} \\ \chi_2 &:= \frac{2t^2(1+3t^2+3t^4+t^6)+2t(1+2t^2+2t^4+2t^6+t^8)}{(1-t^4)(1-t^6)}.\end{aligned}$$

Then we have the following identity

$$\chi_0 + \chi_2 - \chi_1 = \frac{p}{(1-t^4)(1-t^6)}$$

with

$$\begin{aligned}p &= (1+t^3)^2(1+t^5)^2 + 2t^2(1+3t^2+3t^4+t^6) \\ &\quad + 2t(1+2t^2+2t^4+2t^6+t^8) - (1+t)^2(1+t^3)^2(1+t^2+t^4) \\ &= 2t^3 + t^6 - 2t^7 - 2t^9 - t^{10} - t^{12} + 2t^{13} + t^{16} = (2t^3 + t^6)(1-t^4)(1-t^6)\end{aligned}$$

and therefore

$$(3.15) \quad \chi_0 + \chi_2 = \chi_1 + (2t^3 + t^6).$$

This together with the fact that $\mathbb{F}_2[v_2, v_3] \otimes \Lambda(e_3, e'_3, e_5, e'_5)$ is a subalgebra of $H^*(C_\Gamma(E_1); \mathbb{F}_2)$ already implies that the sequence

$$0 \rightarrow \mathbb{F}_2[v_2, v_3] \otimes \Lambda(d_3, d'_3, d_5, d'_5) \rightarrow H^*(C_\Gamma(E_1); \mathbb{F}_2) \rightarrow \text{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}(St, H^*(C_\Gamma(E_1); \mathbb{F}_2)) \rightarrow 0$$

is exact except possibly in dimensions 3 and 6.

In order to complete the proof of a) it is now enough to verify that the inverse limit agrees with $\mathbb{F}_2[v_2, v_3] \otimes \Lambda(d_3, d'_3, d_5, d'_5)$ in degrees 3 and 6. For this we need to analyze the map φ in the exact sequence (3.9). We leave this straightforward verification to the reader.

Then b) follows immediately from (a) together with (3.15) and the exact sequence (3.9), and (c) follows from Theorem 2.1 and the fact that $r_2(G) = 2$. \square

We can now give the proof of Theorem 1.2.

Proof. a) Let X^{E_1} be the fixed points for the action of E_1 on X . The map

$$\Gamma \times_{C_\Gamma(E_1)} X^{E_1} \rightarrow X_s, \quad (g, x) \mapsto gx$$

followed by the inclusion $X_s \subset X$ induces in Borel cohomology the map

$$H^*(\Gamma; \mathbb{F}_2) = H_\Gamma^*(X; \mathbb{F}_2) \rightarrow H_\Gamma^*(\Gamma \times_{C_\Gamma(E_1)} X^{E_1}; \mathbb{F}_2) \cong H_{C_\Gamma(E_1)}^*(X^{E_1}; \mathbb{F}_2) \cong H^*(C_\Gamma(E_1); \mathbb{F}_2)$$

which is induced by restriction. Here we note that the last isomorphism follows from classical Smith theory which guarantees that X^{E_1} is mod 2-acyclic if X is mod-2 acyclic. The claim now follows from the exact sequences (2.1) and (3.9) which show that the epimorphism of Theorem 1.1 is given by this map if we identify the inverse limit with a subalgebra of $H^*(C_\Gamma(E_1); \mathbb{F}_2)$.

b) The two ring homomorphisms $\pi_k : \mathbb{Z}[\frac{1}{2}, i] \rightarrow \mathbb{F}_5$ of (3.3) determine homomorphisms $SL_3(\mathbb{Z}[\frac{1}{2}, i]) \subset GL_3(\mathbb{Z}[\frac{1}{2}, i]) \rightarrow GL_3(\mathbb{F}_5)$. By [Q2] we have

$$H^*GL_3(\mathbb{F}_5; \mathbb{F}_2) \cong \mathbb{F}_3[c_1, c_2, c_3] \otimes \Lambda(q_1, q_3, q_5)$$

and (again by abuse of notation) we define φ via $\varphi(c_i) = \pi_1^*(c_i)$ for $i = 2, 3$, $\varphi(e_i) = \pi_1^*(q_i)$ and $\varphi(e'_i) = \pi_2^*(e'_i)$ for $i = 3, 5$. (Note that because we do not know whether the restriction homomorphism from $H^*GL_3(\mathbb{Z}[\frac{1}{2}, i]; \mathbb{F}_2)$ to the cohomology of the subgroup of diagonal

matrices is injective we make a choice in the definition of $\varphi(c_i)$. For the definition of the classes q_i we refer to section 5 below.

In order to determine the image of these classes with respect to ψ we can calculate at the level of \mathbb{F}_5 and use naturality with respect to the homomorphisms induced by π_1 and π_2 . In fact, the inclusions

$$GL_1(\mathbb{F}_5) \times GL_1(\mathbb{F}_5) \rightarrow GL_2(\mathbb{F}_5) \subset SL_3(\mathbb{F}_5) \rightarrow GL_3(\mathbb{F}_5)$$

induce in cohomology a map

$$\mathbb{F}_3[c_1, c_2, c_3] \otimes \Lambda(q_1, q_3, q_5) \rightarrow \mathbb{F}_2[y_1, y_2] \otimes \Lambda(x_1, x_2)$$

which is easily seen to be given by

$$\begin{aligned} c_1 &\mapsto 0, & c_2 &\mapsto y_1^2 + y_1 y_2 + y_2^2, & c_3 &\mapsto y_1 y_2 (y_1 + y_2) \\ q_1 &\mapsto 0, & q_3 &\mapsto y_1 x_2 + y_2 x_1, & q_5 &\mapsto y_1^2 x_2 + y_2^2 x_1. \end{aligned}$$

On the other hand by (3.4) we know the restriction map induced by the inclusion $GL_1(\mathbb{F}_5) \times GL_1(\mathbb{F}_5) \rightarrow GL_2(\mathbb{F}_5)$ and by part a) of Proposition 3.3 it is now straightforward to check that we have $\psi(\phi(c_i)) = v_i$ for $i = 2, 3$, $\psi(\phi(e_i)) = d_i$ and $\psi(\phi(e'_i)) = d'_i$ for $i = 3, 5$.

c) The space X can be taken to be the product of symmetric space $X_\infty := SL_3(\mathbb{C})/SU(2)$ and the Bruhat-Tits building X_2 for $SL_3(\mathbb{Q}_2[i])$. Now $SL_3(\mathbb{Q}_2[i]) \backslash X_2$ is a 2-simplex (cf. [B]) and the projection map $X \rightarrow X_2$ induces a map

$$SL_3(\mathbb{Q}_2[i]) \backslash X \rightarrow SL_3(\mathbb{Q}_2[i]) \backslash X_2$$

whose fibres have the homotopy type of a 6-dimensional $SL_3(\mathbb{Z}[\frac{1}{2}, i])$ -invariant deformation retract (cf. section 4). Therefore we get $H_G^n(X, X_s; \mathbb{F}_2) = 0$ if $n > 8$ and the inclusion $X_s \subset X$ induces an isomorphism $H_G^n(X; \mathbb{F}_2) \cong H_G^n(X_s; \mathbb{F}_2)$ if $n > 8$. Then part c) simply follows from b) except for the finiteness statement for the kernel for which we refer to (4.1) and (4.2) below. \square

4. COMMENTS ON STEP 2

The situation for $p = 2$ and $G = SL_3(\mathbb{Z}[\frac{1}{2}, i])$ is analogous to the situation for $p = 2$ and $G = SL_3(\mathbb{Z}[\frac{1}{2}])$ for which step 2 was carried out in [H2] via a detailed study of the relative cohomology $H_G^*(X, X_s; \mathbb{F}_2)$ for X equal to the product of the symmetric space $X_\infty := SL_3(\mathbb{R})/SO(3)$ with the Bruhat-Tits building X_2 for $SL_3(\mathbb{Q}_2)$; the spaces involved had a few hundred cells and the calculation was painful. In the case of $SL_3(\mathbb{Z}[\frac{1}{2}, i])$ with X the product of $SL_3(\mathbb{C})/SU(3)$ with the Bruhat-Tits building for $SL_3(\mathbb{Q}_2[i])$ the calculational complexity of the second step is much more involved and an explicit calculation by hand does not look feasible. However, in recent years there have been a lot of machine aided calculations of cohomology of arithmetic groups (for example [GG], [BRW]) and a machine aided calculation seems to be within reach.

The natural strategy for undertaking this second step is to follow the same path as in [H2]. The equivariant cohomology $H_\Gamma^*(X, X_s; \mathbb{F}_2)$ can be studied via the spectral sequence of the projection map

$$p : X = X_\infty \times X_2 \rightarrow X_2.$$

This gives a spectral sequence with

$$(4.1) \quad E_1^{s,t} \cong \bigoplus_{\sigma \in \Lambda_s} H_{\Gamma_\sigma}^t(X_\infty, X_{\infty, s}(\sigma); \mathbb{F}_2) \implies H_\Gamma^{s+t}(X, X_s; \mathbb{F}_2).$$

Here Λ_s indexes the s -dimensional cells in the orbit space of X_2 with respect to the action of Γ . The orbit space is a 2-simplex, i.e. Λ_0 and Λ_1 contain 3 elements and Λ_2 is a singleton. Furthermore Γ_σ is the isotropy group of a chosen representative in X_2 of the cell σ in the

quotient space. For fixed s all s -dimensional cells have isomorphic isotropy groups because the Γ -action on the Bruhat-Tits building is the restriction of a natural action of $GL_3(\mathbb{Z}[\frac{1}{2}, i])$ on X_2 which is transitive on the set of s -dimensional cells (cf. [B]).

Therefore all isotropy subgroups for the action on X_2 are, up to isomorphism, subgroups of $SL_3(\mathbb{Z}[i])$ which itself appears as isotropy group of a 0-dimensional cell in X_2 . By the Soulé-Lannes method the fibre X_∞ of the projection map p admits a 6-dimensional $SL_3(\mathbb{Z}[i])$ -equivariant deformation retract (the space of “well-rounded hermitean forms” modulo arithmetic equivalence) with compact quotient (cf. [Ash]) and therefore we have

$$(4.2) \quad E_1^{s,t} = 0 \text{ unless } s = 0, 1, 2, \ 0 \leq t \leq 6, \text{ and } \dim_{\mathbb{F}_2} E_1^{s,t} < \infty \text{ for all } (s, t) .$$

The E_1 -term of this spectral sequence should be accessible to machine calculation. The spectral sequence will necessarily degenerate at E_3 and the calculation of the d_1 -differential and, if necessary the d_2 -differential, is likely to need human intervention, as it was necessary in the case of $SL(3, \mathbb{Z}[\frac{1}{2}])$ (cf. section 3.4 of [H2]). Likewise the calculation of the connecting homomorphism for the mod-2 Borel cohomology of the pair (X, X_s) is likely to require human intervention.

5. RELATION TO QUILLEN’S CONJECTURE

A major motivation for studying the mod-2 cohomology of $SL_3(\mathbb{Z}[\frac{1}{2}, i])$ comes from a conjecture of Quillen (Conjecture 14.7 of [Q1]) which concerns the structure of the mod p -cohomology of $GL_n(\Lambda)$ where Λ is a ring of S -integers in a number field such that p is invertible in Λ and Λ contains a primitive p -th root of unity ζ_p .

The conjecture stipulates that under these assumptions $H^*GL_n(\Lambda; \mathbb{Z}/p)$ is free over $\mathbb{Z}/p[c_1, \dots, c_n]$ where the c_i are the mod- p Chern classes associated to an embedding of Λ into the complex numbers. In the sequel we will denote this conjecture by $C(n, \Lambda, p)$.

From now on we assume $p = 2$ and $\Lambda = \mathbb{Z}[\frac{1}{2}, i]$. The assumptions on $C(n, \Lambda, 2)$ are clearly satisfied in this case.

Proposition 5.1. *Suppose $n \geq 2$. Then the following statements are equivalent.*

- a) $C(n, \mathbb{Z}[\frac{1}{2}, i], n)$ holds.
- b) The restriction homomorphism $H^*(GL_n(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2) \rightarrow H^*(D_n(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2)$ is injective where $D_n(\mathbb{Z}[\frac{1}{2}, i])$ denotes the subgroup of diagonal matrices in $GL_n(\mathbb{Z}[\frac{1}{2}, i])$.
- c) There are isomorphisms

$$H^*(GL_n(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2) \cong \mathbb{F}_2[c_1, \dots, c_n] \otimes \Lambda(e_1, e'_1, \dots, e_{2n-1}, e'_{2n-1})$$

where the classes c_k are the Chern classes of the tautological n -dimensional complex representation of $GL_n(\mathbb{Z}[\frac{1}{2}, i])$ and the classes e_{2k-1}, e'_{2k-1} are of cohomological degree $2k-1$ for $k = 1, \dots, n$ which will be introduced in (5.1) below.

Proof. It is trivial that (c) implies (a).

In order to show that (a) implies (b) we observe that $D_n(\mathbb{Z}[\frac{1}{2}, i])$ is the centralizer of the unique, up to conjugacy, maximal elementary abelian 2-subgroup E_n of $GL_n(\mathbb{Z}[\frac{1}{2}, i])$ given by the diagonal matrices of order 2. Now consider the top Dickson invariant ω in $H^*(BGL_n(\mathbb{C}); \mathbb{F}_2)$, i.e. the class whose restriction to $H^*B(\prod_{i=1}^n GL_1(\mathbb{C})); \mathbb{F}_2)$ is the product of all non-trivial classes of degree 2. The image of ω in $H^*(GL_n(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2)$ restricts trivially

to the cohomology of all elementary abelian 2- subgroups E of $GL_n(\mathbb{Z}[\frac{1}{2}, i])$ of rank less than n . If (a) holds then the image of ω is not a zero divisor in $H^*(GL_n(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2)$ and hence Corollary 5.8 of [HLS] implies that the restriction to the centralizer of E_n is injective.

The implication (b) \Rightarrow (c) follows from Proposition 5.4 below. \square

Remark 5.2. Conjecture $C(n, \mathbb{Z}[\frac{1}{2}, i], 2)$ is trivially true for $n = 1$ and has been verified for $n = 2$ in [W]. On the other hand, Dwyer's method in [D] using étale approximations X_n for the homotopy type of the 2-completion of $BGL_n(\mathbb{Z}[\frac{1}{2}, i])$ and comparing the set of homotopy classes of $[BP, X_n]$ with that of $[BP, BGL_n(\mathbb{Z}[\frac{1}{2}, i])]$ can be adapted to disprove $C(16, \mathbb{Z}[\frac{1}{2}, i], 2)$. We will not dwell on this in this paper because we will focus on more elementary methods avoiding étale homotopy theory. However, we note that étale approximations can also be used to show that if $C(3, \mathbb{Z}[\frac{1}{2}, i], 2)$ fails then $C(6, \mathbb{Z}[\frac{1}{2}, i], 2)$ fails as well.

Before we go on we introduce the classes e_{2k-1} and e'_{2k-1} . As in the case of GL_2 they are obtained from Quillen's classes $q_{2k-1} \in H^{2k-1}(GL_n(\mathbb{F}_5); \mathbb{F}_2)$ [Q2] which restrict in the cohomology of diagonal matrices in \mathbb{F}_5 to the symmetrization of the class $y_1 \dots y_{k-1} x_k$ where y_k is of cohomological degree 2 corresponding to the k -th factor in the product $\prod_{k=1}^n \mathbb{F}_5^\times$ and x_k is of cohomological degree 1 of the same factor. Then we define

$$(5.1) \quad e_{2k-1} := \pi_1^*(q_{2k-1}), \quad e'_{2k-1} := \pi_2^*(q_{2k-1})$$

where π_1, π_2 are the two ring homomorphisms $\mathbb{Z}[\frac{1}{2}, i] \rightarrow \mathbb{F}_5$ with π_1 sending i to 3 and π_2 sending i to 2 which we considered earlier in section 3. If we identify the mod-2 cohomology $H^*(D_n(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2)$ with $\mathbb{F}_2[y_1, \dots, y_n] \otimes \Lambda(x_1, x'_1, \dots, x_n, x'_n)$ with $y_k, k = 1, \dots, n$ of degree 2 and $x_k, x'_k, k = 1, \dots, n$ of degree 1 where as before we choose x_k and x'_k to be the basis which is dual to the basis of the k -th factor in

$$D_n(\mathbb{Z}[\frac{1}{2}, i])/D_n(\mathbb{Z}[\frac{1}{2}, i])^2 \cong \left(\mathbb{Z}[\frac{1}{2}, i]^\times / (\mathbb{Z}[\frac{1}{2}, i]^\times)^2 \right)^n$$

given by the classes of i and $1 + i$ then we get the following lemma whose straightforward proof we leave to the reader.

Lemma 5.3. *The class e_{2k-1} restricts in the cohomology of the subgroups of diagonal matrices $H^*(D_n(\mathbb{Z}[\frac{1}{2}, i]; \mathbb{F}_2))$ to the symmetrization of $y_1 \dots y_{k-1} x_k$ and the class e'_{2k-1} restricts to the symmetrization of $y_1 \dots y_{k-1} (x_k + x'_k)$. \square*

The following result determines the image of the restriction homomorphism and shows that (b) implies (c) in Proposition 5.1. It resembles results of Mitchell [M] for $GL_n(\mathbb{Z}[\frac{1}{2}])$ for $p = 2$ and of Anton [An1] for $GL_n(\mathbb{Z}[\frac{1}{3}, \zeta_3])$ for $p = 3$.

Proposition 5.4. *Let $n \geq 1$ be an integer. The image of the restriction map*

$$i^* : H^*(GL_n(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2) \rightarrow H^*(D_n(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2) \cong \mathbb{F}_2[y_1, \dots, y_n] \otimes \Lambda(x_1, x'_1, \dots, x_n, x'_n)$$

is isomorphic to

$$\mathbb{F}_2[c_1, \dots, c_n] \otimes \Lambda(e_1, e'_1, \dots, e_{2n-1}, e'_{2n-1}) .$$

Proof. The case $n = 1$ is trivial and for $n = 2$ this follows from Theorem 1 of [W] (cf. (3.2) and (3.4)) and from Lemma 5.3. Furthermore Lemma 5.3 shows that the image of the map

$$H^*(GL_n(\mathbb{F}_5 \times \mathbb{F}_5); \mathbb{F}_2) \xrightarrow{(\pi_1^*, \pi_2^*)} H^*(GL_n(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2) \xrightarrow{i^*} H^*(D_n(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2)$$

agrees with

$$\mathbb{F}_2[c_1, \dots, c_n] \otimes \Lambda(e_1, e'_1, \dots, e_{2n-1}, e'_{2n-1}) .$$

Therefore the proposition is an immediate consequence of Proposition 5.5 below. \square

We note that matrix block sum determines associative algebra structures on the mod-2 homology of the spaces

$$\coprod_{n \geq 0} BD_n(\mathbb{Z}[\frac{1}{2}, i]), \quad \coprod_{n \geq 0} BGL_n(\mathbb{Z}[\frac{1}{2}, i]), \quad \coprod_{n \geq 0} BGL_n(\mathbb{F}_5 \times \mathbb{F}_5)$$

such that the inclusions $D_n(\mathbb{Z}[\frac{1}{2}, i]) \rightarrow GL_n(\mathbb{Z}[\frac{1}{2}, i])$ and the homomorphisms $\pi_1 \times \pi_2 : \mathbb{Z}[\frac{1}{2}, i] \rightarrow \mathbb{F}_5 \times \mathbb{F}_5$ determine homomorphisms of bigraded algebras

$$\bigoplus_{n \geq 0} H_*(D_n(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2) \xrightarrow{i_*} \bigoplus_{n \geq 0} H_*(GL_n(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2)$$

and

$$\bigoplus_{n \geq 0} H_*(GL_n(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2) \xrightarrow{(\pi_1, \pi_2)^*} \bigoplus_{n \geq 0} H_*(GL_n(\mathbb{F}_5 \times \mathbb{F}_5); \mathbb{F}_2) .$$

Proposition 5.5. *The kernel of the map*

$$\bigoplus_{n \geq 0} H_*(D_n(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2) \xrightarrow{i_*} \bigoplus_{n \geq 0} H_*(GL_n(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2) \xrightarrow{(\pi_1, \pi_2)^*} \bigoplus_{n \geq 0} H_*(GL_n(\mathbb{F}_5 \times \mathbb{F}_5); \mathbb{F}_2)$$

is the homogeneous ideal generated by the kernel of the map

$$H_*(D_2(\mathbb{Z}[\frac{1}{2}, i]; \mathbb{F}_2)) \rightarrow H_*(GL_2(\mathbb{F}_5 \times \mathbb{F}_5); \mathbb{F}_2) .$$

This result is implicit in Proposition 3.6 of [An2]. For the convenience of the reader we give a short outline of a slightly modified proof. The ideas are due to [An2].

Proof. The monomial basis of

$$H^*(D_n(\mathbb{Z}[\frac{1}{2}, i]; \mathbb{F}_2)) \cong \mathbb{F}_2[y_1, \dots, y_n] \otimes \Lambda(x_1, x'_1, \dots, x_n, x'_n)$$

can be indexed by the set $S(n)$ of sequences $I = (k_1, \dots, k_n, \varepsilon_{1,1}, \dots, \varepsilon_{1,n}, \varepsilon_{2,1}, \dots, \varepsilon_{2,n})$ where the k_i are integers ≥ 0 and $\varepsilon_{i,j} \in \{0, 1\}$ for $i = 1, 2$ and $1 \leq j \leq n$. More precisely to I we associate the monomial

$$y^I := y_1^{k_1} \dots y_n^{k_n} x_1^{\varepsilon_{1,1}} \dots x_n^{\varepsilon_{1,n}} x'_1{}^{\varepsilon_{2,1}} \dots x'_n{}^{\varepsilon_{2,n}} .$$

Likewise the monomial basis in the subalgebra Q_n of

$$H^*(GL_n(\mathbb{F}_5) \times GL_n(\mathbb{F}_5); \mathbb{F}_2) \cong H^*(GL_n(\mathbb{F}_5); \mathbb{F}_2) \otimes H^*(GL_n(\mathbb{F}_5); \mathbb{F}_2)$$

generated by the classes $c_i \otimes 1, q_{2i-1} \otimes 1, 1 \otimes q_{2i-1}, i = 1, \dots, n$, can be indexed by the same set of sequences which we prefer, however, to denote $T(n)$. Here we associate to I the monomial

$$c^I := (c_1 \otimes 1)^{k_1} \dots (c_n \otimes 1)^{k_n} (q_1 \otimes 1)^{\varepsilon_{1,1}} \dots (q_n \otimes 1)^{\varepsilon_{1,n}} (1 \otimes q_1)^{\varepsilon_{2,1}} \dots (1 \otimes q_n)^{\varepsilon_{2,n}} .$$

Here we have made a choice in privileging the classes $c_i \otimes 1$ over $1 \otimes c_i$, but this choice is not relevant for the sequel. We observe that both $T(n)$ and $S(n)$ can be and in the sequel will be ordered lexicographically.

If we denote the map in the statement of Proposition 5.5 by t_* , its dual by t^* and the dual basis of the monomial basis y^I of $H^*(D_n(\mathbb{Z}[\frac{1}{2}, i]; \mathbb{F}_2))$ by u_I then we have for every $K \in T(n)$

$$t^*(c^K) = \sum_{I \in S(n)} [K : I] y^I$$

for unique elements $[K : I] \in \mathbb{F}_2$ which satisfy

$$[K : I] = \langle t^*(c^K), u_I \rangle = \langle c^K, t_* u_I \rangle$$

where $\langle -, - \rangle$ denotes the Kronecker product between cohomology and homology. Next we consider the map

$$\alpha : T(n) \rightarrow S(n)$$

which sends

$$K = (a_1, \dots, a_n, \varepsilon_{1,1}, \dots, \varepsilon_{1,n}, \varepsilon_{2,1}, \dots, \varepsilon_{2,n}) \in T(n)$$

to the maximal $I \in S(n)$ such that $[K : I] \neq 0$. By using that c_i restricts to the symmetrization of $y_1 \dots y_i$, that e_{2k-1} restricts to the symmetrization of $y_1 \dots y_{k-1} x_k$ and e'_{2k-1} to the symmetrization of $y_1 \dots y_{k-1} (x_k + x'_k)$ we see that $\alpha(K)$ is given by the sequence

$$\alpha(K) = (k_1, \dots, k_n, \phi_{1,1}, \dots, \phi_{1,n}, \phi_{2,1}, \dots, \phi_{2,n})$$

with

$$\begin{aligned} k_1 &= a_1 + \dots a_n + (\phi_{1,2} + \dots \phi_{1,n}) \\ k_2 &= a_2 + \dots a_n + (\phi_{2,3} + \dots \phi_{2,n}) \\ &\dots \dots \\ k_j &= a_j + \dots a_n + (\phi_{j,j+1} + \dots \phi_{j,n}) \\ &\dots \dots \\ k_n &= a_n \\ \phi_{i,j} &= \varepsilon_{i,j}, \quad 1 \leq j \leq n, i = 1, 2. \end{aligned}$$

From these identities it is obvious that α is injective and I is in the image of α if and only if we have

$$(5.2) \quad k_j - k_{j-1} \geq \phi_{1,j+1} + \phi_{2,j+1} \quad \text{for all } 1 \leq j \leq n.$$

Furthermore, if we define for $I \in S(n)$ its degree

$$\deg(I) = \sum_{j=1}^n (k_j + \sum_{i=1}^r \varepsilon_{i,j})$$

then

$$(5.3) \quad [K : I] = 0 \quad \text{if } \deg(I) \neq \deg(\alpha(K)).$$

In fact, all monomials y^I occurring with nonzero coefficients in the expansion of c^K have the same degree and $\alpha(K)$ is one of these monomials.

Now let $x \in H_*(D_n(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2)$ be a homogeneous element in the kernel of t and suppose $x \neq 0$. Then we can write x as a linear combination in the dual basis u_I , i.e.

$$x = \sum_{I \in S(n)} (x : I) u_I$$

for unique elements $(x : I) \in \mathbb{F}_2$ and $(x : I) = 0$ for all but a finite number of I . Iterated application of the following lemma then shows that x is in the ideal generated by $\text{Ker}(t_2)$. \square

Lemma 5.6. *Suppose $n \geq 2$ and let $x \in H_*(D_n(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2)$ be a non-trivial homogeneous element in the kernel of t_n and let I_0 be the unique minimal sequence among those $J \in S(n)$ with $(x : J) \neq 0$. Then there exists a homogeneous element $y = \sum_{I \in S(n)} (y : I) u_I$ in the ideal generated by the kernel of $(t_2)_*$ such that*

$$a) \quad (y : I_0) = (x : I_0)$$

$$b) \quad (y : I) = (x : I) \text{ for } I < I_0.$$

Proof. First we note that I_0 cannot be in the image of α . In fact, if $K \in T(n)$ then

$$(5.4) \quad 0 = \langle c^K, t_*(x) \rangle = \sum_{I \in S(n)} (x : I) \langle c^K, u_I \rangle = \sum_{I \in S(n)} [K : I] (x : I)$$

and in this last sum we have $[K : I] = 0$ if $I > \alpha(K)$ and $[K : \alpha(K)] = 1$ by definition of $\alpha(K)$. If there exists K such that $\alpha(K) = I_0$ then $(x : J) = 0$ for all $J < I_0$ and (5.4) simplifies to $0 = [K : I_0] (x : I_0) = (x : I_0)$ in contradiction to the defining property of I_0 .

Because I_0 is not in the image of α we see from (5.2) that there exists j with $1 \leq j \leq n$ and $I_1 \in S(j-1)$, $I_2 \in S(2)$ and $I_3 \in S(n-j-1)$ with I_2 violating condition (5.2) for this j such that $I_0 = I_1 I_2 I_3$ is the concatenation of I_1 , I_2 and I_3 . Here the concatenation of two sequences $I = (k_1, \dots, k_r, \delta_{1,1}, \dots, \delta_{1,r}, \delta_{2,1}, \dots, \delta_{2,r}) \in S(r)$ and $J = (l_1, \dots, l_s, \varepsilon_{1,1}, \dots, \varepsilon_{1,s}, \varepsilon_{2,1}, \dots, \varepsilon_{2,s}) \in S(s)$ is defined to be the sequence $IJ \in S(r+s)$ given by

$$IJ = (k_1, \dots, k_r, l_1, \dots, l_s, \delta_{1,1}, \dots, \delta_{1,r}, \varepsilon_{1,1}, \dots, \varepsilon_{1,s}, \delta_{2,1}, \dots, \delta_{2,r}, \varepsilon_{2,1}, \dots, \varepsilon_{2,s}) .$$

By the definition of concatenation of sequences we have $u_{IJ} = u_I u_J$.

Therefore, if $(t_2)_*(u_{I_2}) = 0$ then $u_{I_0} = u_{I_1} u_{I_2} u_{I_3}$ and $y = u_{I_0}$ is the desired element. If $(t_2)_*(u_{I_2}) \neq 0$ we write

$$(t_2)_*(u_{I_2}) = \sum_{K \in T(2)} [K : I_2] v_K$$

with v_K the dual basis to the monomial basis c^K , $K \in T(2)$. This is possible because we already know that Proposition 5.4 is true for $n = 2$, i.e. the image of $(t_2)_*$ is dual to the vector space spanned by the c_K with $K \in T(2)$. Let us denote the set of those $K \in T(2)$ with $\deg(\alpha(K)) = \deg(I_2)$ by T . Then T is finite because α is injective and because there are only finitely many sequences in $S(n)$ with a fixed degree. Hence the set $\alpha(T)$ has a minimum, say $\alpha(K_0)$. Because I_2 violates (5.2) it is not in the image of α and therefore by the defining property of $\alpha(K_0)$ we must have $I_2 < \alpha(K_0)$.

Then consider for each $K \in T$ the following linear equation

$$0 = [K : I_2] + \sum_{J \in T} [K : \alpha(J)] \lambda_J$$

for unknowns $(\lambda_J)_{J \in T}$. By definition of α we have $[K : \alpha(K)] = 1$ and $[K : \alpha(J)] = 0$ if $\alpha(J) < \alpha(K)$. Therefore, if $|T|$ denotes the cardinal of T these $|T|$ equations for $|T|$ unknowns form a linear system of equations of triangular shape with 1's on the diagonal. Hence there exists a (unique) solution $(\lambda_J)_{J \in T}$. Furthermore if $K \in T(2)$ does not belong to T then by (5.3) we have $[K : \alpha(J)] = 0$ and $[K : I_2] = 0$ for all $J \in T$ and hence

$$w := u_{I_2} + \sum_{J \in T} \lambda_J u_{\alpha(J)}$$

satisfies

$$(t_2)_*(w) = \sum_{K \in T(2)} ([K : I_2] + \sum_{J \in T} \lambda_J [K : \alpha(J)]) v_K = \sum_{K \in T} ([K : I_2] + \sum_{J \in T} \lambda_J [K : \alpha(J)]) v_K = 0$$

and $y = u_{I_1} w u_{I_3}$ is the desired element. \square

Finally we relate $C(3, \mathbb{Z}[\frac{1}{2}, i], n)$ to the behaviour of the restriction homomorphism

$$H^*(\Gamma; \mathbb{F}_2) \rightarrow H^*(C_\Gamma(E_2); \mathbb{F}_2) .$$

For this we observe that the subgroups $\Gamma = SL_3(\mathbb{Z}[\frac{1}{2}, i])$ and the center $Z \cong \mathbb{Z}[\frac{1}{2}, i]^\times$ of $GL_3(\mathbb{Z}[\frac{1}{2}, i])$ have trivial intersection and their product is the kernel of the homomorphism

$$GL_3(\mathbb{Z}[\frac{1}{2}, i]) \rightarrow GL_1(\mathbb{Z}[\frac{1}{2}, i]) / (GL_1(\mathbb{Z}[\frac{1}{2}, i]))^3 \cong \mathbb{Z}/3$$

which is induced by the determinant. Therefore

$$(5.5) \quad H^*(GL_3(\mathbb{Z}[\frac{1}{2}, i]; \mathbb{F}_2) \cong (H^*(SL_3(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2) \otimes H^*(Z; \mathbb{F}_2))^{\mathbb{Z}/3}.$$

Corollary 5.7. *If $C(3, \mathbb{Z}[\frac{1}{2}, i], 2)$ is true then either*

a) $H^(SL_3(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2) \cong \mathbb{F}_2[v_2, v_3] \otimes \Lambda(d_3, d'_3, d_5, d'_5)$ or*

b) the kernel of the map ψ of Theorem 1.2 is a finite dimensional vector space on which $\mathbb{Z}/3 \cong GL_3(\mathbb{Z}[\frac{1}{2}, i]) / (GL_3(\mathbb{Z}[\frac{1}{2}, i]))^3$ acts without invariants.

Proof. The quotient $\mathbb{Z}/3 \cong GL_3(\mathbb{Z}[\frac{1}{2}, i]) / (GL_3(\mathbb{Z}[\frac{1}{2}, i]))^3$ acts clearly trivially on $H^*(Z; \mathbb{F}_2)$ and on the subalgebra of $H^*(SL_3(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2)$ generated by $v_2, v_3, d_3, d'_3, d_5, d'_5$ and hence the corollary follows immediately from (5.5) and Theorem 1.2. \square

REFERENCES

- [An1] M. Anton, *On a conjecture of Quillen at the prime 3*, Journal of Pure and Applied Algebra **144** (1999) 1-20
- [An2] M. Anton, *An elementary invariant problem and general linear group cohomology restricted to the diagonal subgroup*, Trans. Amer. Math. Soc. **335** (2003), 2327–2340
- [Ash] A.Ash, *Small-dimensional classifying spaces for arithmetic subgroups of general linear groups*, Duke Math. Journal **51** (1984), 459–468
- [B] K. Brown, *Buildings*, Springer Verlag 1989
- [BRW] T.A. Bui, A. D. Rahm and M. Wendt, *The Farrell Tate and Bredon homology for $PSL_4(\mathbb{Z})$ and other arithmetic groups*, Preprint 2016
- [D] W. G. Dwyer, *Exotic cohomology for $GL_n(\mathbb{Z}[1/2])$* , Proc. Amer. Math. Soc. **126**, 2159–2167
- [GG] H. Gangl, P. E. Gunnells, J. Hanke, A. Schürmann, M. Dutour Sikirić and D. Yasaki, *On the cohomology of linear groups over imaginary quadratic fields*, arXiv 1307.1165 (2016)
- [H1] H.-W. Henn, *Centralizers of elementary abelian p -subgroups, the Borel construction of the singular locus and applications to the cohomology of discrete groups*, Topology **36** (1997), 271-286
- [H2] H.-W. Henn, *The cohomology of $SL(3, \mathbb{Z}[1/2])$* , K-Theory **16** (1999), 299–359
- [HLS] H.-W. Henn, J. Lannes and L. Schwartz, *Localization of unstable A - modules and equivariant mod- p cohomology*, Math. Ann. **301** (1995), 23–68
- [M] S. Mitchell, *On the plus construction for $BGL\mathbb{Z}[\frac{1}{2}]$* , Math.Zeit. **209** (1992), 205–222
- [Q1] D. Quillen, *The spectrum of an equivariant cohomology ring I, II*, Ann. of Math. **94** (1971), 549–572, 573-602
- [Q2] D. Quillen, *On the cohomology and K-theory of the general linear groups over a finite field*, Ann. Math. **96** (1972), 552–586,
- [W] Nicolas Weiss, *Cohomologie de $GL_2(\mathbb{Z}[i, \frac{1}{2}])$ à coefficients dans \mathbb{F}_2* , Thèse à Strasbourg, 2006, available at <https://hal.archives-ouvertes.fr/tel-00174888>